Relativistic Tunneling Through Two Successive Barriers

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We study the relativistic quantum mechanical problem of a Dirac particle tunneling through two successive electrostatic barriers. Our aim is to study the emergence of the so-called *Generalized Hartman Effect*, an effect observed in the context of nonrelativistic tunneling as well as in its electromagnetic counterparts, and which is often associated with the possibility of superluminal velocities in the tunneling process. We discuss the behavior of both the phase (or group) tunneling time and the dwell time, and show that in the limit of opaque barriers the relativistic theory also allows the emergence of the Generalized Hartman Effect. We compare our results with the nonrelativistic ones and discuss their interpretation.

I. INTRODUCTION

The phenomenon of tunneling is one of the most striking and extensively studied consequences of quantum mechanics. Yet, after decades of scrutiny (for reviews see, for example, [1, 2, 3, 4]) it still presents serious conceptual challenges, such as a meaningful definition of tunneling time, that is, the time it takes for a particle to tunnel through a potential barrier.

Several different scales of time associated with the tunneling process have been proposed (see [1]). Among the most prominent ones are the phase time (or group delay time), given by the energy derivative of the phase shift in the transmission (or reflection) amplitude, and the dwell time, which is related to the average time spent by the particle in the region of the potential.

It is well known that for the tunneling of a particle through an opaque barrier the group delay saturates with the width of the barrier, a phenomenon called *Hartmann effect* [5]. Several authors interpret this saturated time as the transit time for the particle to go through the potential, which would imply, as an immediate consequence, the possibility of superluminal (group) velocities for barriers with a sufficiently large spatial extension. Such an interpretation has been in the center of an intense debate in the literature (see, e.g., [4, 6, 7] and references there cited).

Recently an apparently even more paradoxical effect, which became known as *Generalized Hartman Effect*, has been brought to attention, not only in the context of nonrelativistic quantum tunneling, but also in the context of its electromagnetic counterparts. This effect consists in the fact that for tunneling through two potential barriers separated by a distance l the phase time is, in the limit of opaque barriers, independent not only on the barrier widths but also on the spacing *between* them [8] (see also [9, 10]). In fact, Esposito [11] showed that for a system of N barriers the phase time is independent also on the number of barriers. Despite the fact that phase time cannot, in general, be interpreted as a propagation (or transit) time for the particle (or wavepacket), this effect is counterintuitive since one would, *naively*, expect that in the space between the barriers the group delay could be viewed as a propagation time, and therefore, it should depend on the distance between the barriers.

In the last years several papers have also analyzed the problem of quantum tunneling from a relativistic standpoint [12, 13, 14, 15, 16, 17, 18, 19]. Some of these papers were concerned with the fact that the analysis of possible *superluminal* group velocities associated with the Hartmann effect should be properly addressed in the context of a relativistic theory [12, 15, 16, 17, 19]; others were concerned with general aspects of the relativistic problem, such as the relation between phase time and dwell time [18] or the relation between dwell time and Larmor times [14]. In fact, due to the relevance of the tunneling phenomenon, it is important to consider the possible quantitative and/or qualitative differences in the phase (and dwell) time arising due to the relativistic dynamic. What is more, a clear understanding of the relativistic aspects of tunneling is imperative if one wants to eventually obtain a meaningful time scale for this phenomenon, because the instantaneous spread of the probability density in nonrelativistic quantum mechanics [20] makes it difficult to define an unambiguous tunneling time in the context of Schrödinger theory.

Most of the above papers were concerned with a single potential barrier or potential well. In fact, to the best of our knowledge, the only relativistic treatment of

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the two barrier case was due to Leavens and Aers [21], which were concerned with Larmor times at resonance. Thus, the present work complements the previous ones by considering the relativistic approach to the problem of a wave packet tunneling through two successive barriers. We shall address the emergence of the generalized Hartman effect in this context and discuss its possible interpretations. We also compare our results with those obtained in the nonrelativistic framework.

II. THE RELATIVISTIC PHASE AND DWELL TIMES

Here we shall be concerned with the relativistic one-dimensional scattering of a mass m and spin-1/2 wave packet by an electrostatic time-independent potential $V(z)$. The general form of the incident wave packet is given by

$$\Psi(z,t) = \int dE \, A(E)\psi(z)e^{-i\frac{Et}{\hbar}} , \qquad (1)$$

where $A(E)$ are its Fourier coefficients, while $\psi(z)$ must satisfy the time-independent Dirac equation associated with the energy E,

$$\left\{-i\hbar c\alpha^z\partial_z + \beta mc^2 + V(z)\right\}\psi(z) = E\psi(z), \quad (2)$$

with α^z and β being the Dirac matrices (we will follow the conventions in [22]). The potential in which we are interested consists of two potential barriers of height V_0 and width a, and spaced by a distance l, as seen in Figure 1 (since barriers of different heights and widths do not introduce any novelty, they will not be considered here).

Moreover, we shall consider an incident wave packet whose Fourier energy distribution $A(E)$ is very sharply concentrated around a given value E_0, corresponding, therefore, to a smooth modulation of the eigenfunction corresponding to E_0. Such a wave packet must have, therefore, a large spatial extension. For our purposes we shall assume that the energy dispersion of the wavepacket is sufficiently narrow such that its spatial extension is always very large when compared to the extension of the region in which the potential is nonvanishing (the region $0 < z < l + 2a$ in Figure 1). With these assumptions in mind it is justifiable the use of the stationary phase method to follow the position of the peak of the wavepacket in the free regions (regions I and V) (see, for example, [23] and references therein). We shall also consider that E_0 is a positive energy (particle) in the evanescent region, $E_0 - mc^2 < V_0 < E_0 + mc^2$. Therefore, the region of supercritical potential, in which there is pair production (and the associated Klein paradox) and where, therefore, the one-particle Dirac equation ceases to be valid, will not be considered (for a study of the supercritical region for the one barrier potential see [24]).

Considering, as usual, a wave packet incident only from the left and having spin up (we make this assumption
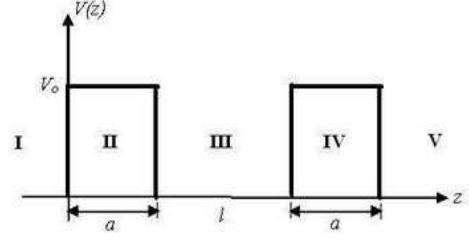

FIG. 1: Schematic view of the two barriers arrangement. We consider two rectangular identical barriers of heigth V_0 and width a, separated between themselves by a distance L.

without loss of generality, because the potential considered here causes no spin flip), the general solution of the stationary problem in the various regions indicated in Figure 1 is given by

$$\psi_I(z) = e^{ikz}u_E(k) + Re^{-ikz}u_E(-k),$$

$$\psi_{II}(z) = Ae^{-qz}u_{E-V_o}(iq) + Be^{qz}u_{E-V_o}(-iq),$$

$$\psi_{III}(z) = Ce^{ikz}u_E(k) + De^{-ikz}u_E(-k), \qquad (3)$$

$$\psi_{IV}(z) = Fe^{-qz}u_{E-V_o}(iq) + Ge^{qz}u_{E-V_o}(-iq),$$

$$\psi_V(z) = Te^{ikz}u_E(k),$$

where

$$u_E(k) = \begin{pmatrix} 1 \\ 0 \\ \frac{ck\hbar}{E+mc^2} \\ 0 \end{pmatrix} \qquad (4)$$

and

$$k \equiv \frac{1}{\hbar c}\sqrt{E^2 - m^2c^4}; \qquad q \equiv \frac{1}{\hbar c}\sqrt{m^2c^4 - (E - V_0)^2}. \qquad (5)$$

The above coefficients can be determined, as usual, from the boundary conditions requiring the wave function to be continuous at the potential discontinuities. After some simple but tedious algebra we obtain the transmission and the reflection amplitudes:

$$T = e^{-2ika}\left\{\left[\cosh(qa) + i\frac{(1-\alpha^2)}{2\alpha}\sinh(qa)\right]^2 + \frac{(1+\alpha^2)^2}{4\alpha^2}\sinh^2(qa)e^{2ikl}\right\}^{-1} \qquad (6)$$

and

$$R = e^{i[k(2a+l)-\frac{\pi}{2}]} \frac{(1+\alpha^2)}{\alpha} \sinh(qa)$$

$$\times \left\{ \cos(kl)\cosh(qa) + \frac{1}{2\alpha}(1-\alpha^2)\sin(kl)\sinh(qa) \right\}$$

$$\times \ T\,, \tag{7}$$

where we have introduced

$$\alpha \equiv \frac{k}{q} \frac{(E-V_0+mc^2)}{(E+mc^2)}\,. \tag{8}$$

It is convenient to express the transmission and reflection coefficients in terms of their phases as

$$T(E) = |T|e^{i[\varphi_t - k(2a+l)]}\,; \tag{9}$$
$$R(E) = |R|e^{i\varphi_r}\,, \tag{10}$$

where $\varphi_r = \varphi_t - \pi/2$, while φ_t is given by

$$\varphi_t = kl - \tan^{-1}\left\{ \frac{4\alpha(1-\alpha^2)\sinh(2qa) - (1+\alpha^2)^2\sin(2kl)[1-\cosh(2qa)]}{4\alpha^2[1+\cosh(2qa)] + [1-\cosh(2qa)][(1-\alpha^2)^2 - (1+\alpha^2)^2\cos(2kl)]} \right\}\,. \tag{11}$$

Now, accordingly to the stationary phase method, the (extrapolated) transmitted and reflected phase times are given, respectively, as [1]

$$\tau_p^t = \hbar\frac{d\varphi_t}{dE}\bigg|_{E_0}\,, \tag{12}$$

$$\tau_p^r = \hbar\frac{d\varphi_r}{dE}\bigg|_{E_0} = \hbar\frac{d\varphi_t}{dE}\bigg|_{E_0}\,, \tag{13}$$

where we used in these expressions the same central energy E_0 of the incident wave packet, what is justifiable by our previous assumptions about the sharp concentration of the initial wave packet around this energy (this corresponds to the situation in which there is essentially no distortion nor reshaping of the transmitted wave, a condition claimed by several authors as necessary to allow a physical meaning to the group velocity [4, 25]). The above expressions imply the equality of the transmitted and the reflected phase times, as it is always the case for symmetric potentials [26]. From now on we will refer to both these times simply as the *phase time* τ_p. Such a time corresponds to the (extrapolated) instant in which the transmitted and reflected wave packet *peak* appear at $z = 2a+l$ and $z = 0$, respectively.

Now, by using a general relation obtained by Winful *et al.* [18, 27], we can determine the dwell time τ_d, which is a measure of the time spent by the particle in the potential region, without distinction of whether it is finally reflected or transmitted [28, 29]. Such a relation, for symmetric potentials, reads

$$\tau_d = \tau_p - \tau_i\,, \tag{14}$$

where τ_i is the self-interference delay, given by

$$\tau_i = -\frac{m}{\hbar k^2}\mathrm{Im}(R)\,. \tag{15}$$

The explicit expressions obtained for the phase and dwell time from the above definitions are not particularly

illuminating and are presented in the Appendix. Here we will discuss their properties. The limit of one barrier (of width $2a$) is easily obtained by assuming $l = 0$ and it agrees with the results of [15] and [16]. Also the nonrelativistic limit, obtained by making $mc^2 \to \infty$ and $V_0 \ll mc^2$, agrees with the results obtained by [8]. In fact, we can verify these limits directly in the expressions for the amplitudes and the transmission phase above.

III. DISCUSSION AND CONCLUDING REMARKS

Of special interest for us is the limit of opaque barriers, $qa \gg 1$, in which the phase and dwell times become

$$\tau_p = \frac{2\hbar}{1+\alpha^2}\left(\frac{d\alpha}{dE}\right) = \frac{2\alpha}{1+\alpha^2}\frac{(k^2+q^2)}{k^2}\frac{m}{\hbar q^2}\,, \tag{16}$$

$$\tau_d = \frac{2\alpha}{(1+\alpha^2)}\frac{m}{\hbar q^2}\,, \tag{17}$$

where we have used the result that in this limit $\tau_i = 2\alpha/(1+\alpha^2)m/(\hbar k^2)$. From the above expressions it is clear that both the phase and dwell times saturate in the opaque limit, not depending either on the width of the barriers or on the distance l between them. This demonstrates that the *generalized Hartmann effect* also emerges in the context of relativistic quantum mechanics. As a consequence, if we *extrapolate* the concept of the *group* velocity into the potential region, it will be given by $v_g \equiv (2a+l)/\tau_p$. This velocity can be made arbitrarily large, allowing for superluminal *group* velocities for sufficiently large barrier widths a, as we shall see below.

Figure 2A shows the typical behavior of the phase and dwell times in the domain of fully relativistic energies, where we fixed all the relevant parameters, except the barriers width a. We observe that for small values of the width a, both the phase and dwell times are greater

than the free and the light times. As the barrier become thicker, both these times grow slower (in fact, they can even decrease, depending on the value of the other parameters, as it is the case shown in the figure) and they can become smaller than the free and the light times. What is more interesting, the phase and dwell times can become smaller than the light time *even before the saturated regime is obtained*. So, in the domain of fully relativistic energies the group velocity can be superluminal even before arriving at the opaque limit. A similar behavior can be observed also in the one barrier case (obtained by taking $l = 0$). For comparison, we also showed the behavior of the phase time calculated from Schrödinger equation. We see that the nonrelativistic theory predicts a phase time of the same order as the relativistic one, and sharing the same behavior, specially in what concerns the possibility of emerging superluminal group velocities before the saturation. Figure 2B shows the same plots for the energies in the relativistic scales, but now with a greater difference between the energy of the incident packet and the height of the barrier. Figures 2A and 2B show that Dirac theory can predict group velocities which can be *smaller* (Figure 2A) *or greater* (Figure 2B) than those predicted by the Schrödinger theory, being each of these situations determined by the specific choice of values for the parameters, especially for the energies. These results are in agreement with those observed for the single barrier case in reference [16], and explain the origin of the apparent contradictory claims of Krekora *et al.* [15] and Leavens and Aers [21], concerning to whether the relativistic theory predicts group velocities smaller or greater than those predicted by the nonrelativistic one. In Figure 2C we can check the complete agreement of the predictions from both the relativistic and the non-relativistic theories in the scale of low (nonrelativistic) energies.

In Figure 3 we plot the behavior of τ_p and τ_d for fixed a and varying l. Figure 3A shows, as expected, a trend to linear increase with l (except for several resonance peaks) as long as a is not very large. That is, outside the opaque domain both the dwell and phase times *do not saturate* with the barrier separation l. The same behavior is observed for the nonrelativistic time with the same values of the parameters (not shown in the figure). We can also observe the equality between the dwell and phase times at resonance ($R = 0$), as predicted by the relation (14). Again we observe that the phase time off-resonance can be smaller than the light time, even before attaining the saturated regime, which imply superluminal group velocities. As the barrier width increases, the off-resonant phase and dwell times tend to saturate to their values at the opaque limit, but still presenting the resonant peaks, as we can observe in Figure 3B. Finally, it is only when $a \to \infty$ that both these times saturate in such a way that the resonant peaks are no longer observed – the generalized Hartmann effect. Therefore, the results of the relativistic theory reinforce the conclusion by Winful [4] that the generalized Hartmann effect is just
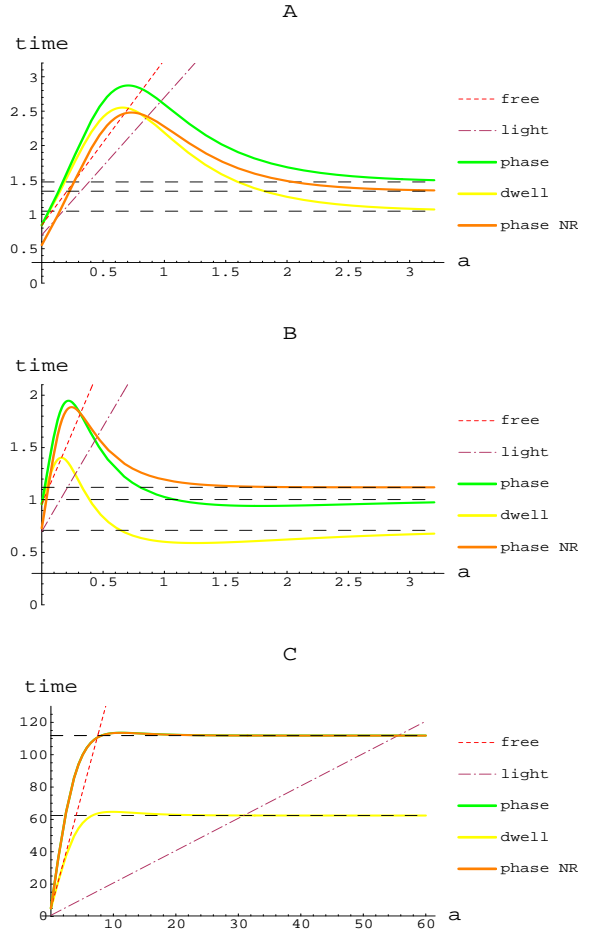

FIG. 2: (Color online) The phase and dwell times *vs* barrier width a. We used the system of *natural units*, in which $\hbar = c = 1$. The energies were expressed in units of the particle rest energy m ($\simeq 0.5$ MeV for an electron, for example). Accordingly, distances and times are given in units of m^{-1}. For reference we also show the *free time* in which the peak of a free wavepacket traverses the region $0 < z < 2a + l$ and the *light time* in which a free light pulse traverses the same region. For comparison we plotted also the nonrelativistic (NR) phase time obtained from the Schrödinger equation [8]. The horizontal dashed lines indicate the corresponding saturated times in the opaque limit. The first two plots correspond to energies in the relativistic scale, while the last one concerns to nonrelativistic energies. In all these plots we take $l = 0.7$. (A) $E_0 = 1.8$, $V_0 = 1.5$; (B) $E_0 = 1.46$, $V_0 = 2.19$; (C) $E_0 = 1.01$, $V_0 = 0.018$ (in this last plot the relativistic and the nonrelativistic phase times coincide).

an artifact resulting from taking the opaque limit before exploring the variation with l (for an alternative argument, using the multiple peaks approach, see [30]).

Let us now look more carefully into the situation characterizing the generalized Hartman effect by evaluating the flux of particles $J^z = c\psi^\dagger \alpha^z \psi$ both in the region between the barriers (as given by ψ_{III}), and to the right

of the potential for the given energy E_0. The solution of the stationary problem gives us

$$C = \left[\cosh(qa) + i\frac{(1-\alpha^2)}{2\alpha}\sinh(qa)\right]e^{ika}T$$

$$D = -i\frac{(1+\alpha^2)}{2\alpha}\sinh(qa)e^{ik(3a+2l)}T,$$

for the coefficients in ψ_{III}, with T being the transmission coefficient. From (6) it is plain that in the opaque limit $T \sim e^{-2qa}$, so that both C and D decay with barrier width as e^{-qa}. Therefore, we conclude that in the opaque limit, $qa \gg 1$, there is essentially no flux, hence, no propagation of particles in regions III and V. Accordingly, it follows that the saturated times would be the same even if the second barrier were absent [18], similarly to what happens in the nonrelativistic theory [10]; in fact, expression (16) is identical to that obtained for the relativistic case of a single barrier at the opaque limit (see [15] and [16]). Thus, the condition to the generalized Hartman effect to occur is the condition of no transmission, in which case it makes no sense to associate any velocity to the tunneling process [4].

On the other hand, it is possible to have situations in which qa is large, but finite, such that there is still an appreciable transmission (before the saturation regime), and the associated group velocities during the tunneling are superluminal. Notice that we have considered wavepackets sharply centered around a given energy in such a way that the transmitted wave packet could be seen essentially as a (attenuated) nondistorted version of the incident one, a feature that is claimed by several authors as allowing one to attribute a physical meaning to the group velocity [25] [4]. Therefore, *if* one maintains the interpretation that the group velocities are propagation velocities, it would seem that relativity does not forbid superluminal tunneling velocities in the single or double barrier tunneling. However, it must be noticed that while there is little doubt that the group velocity in the region V (or region I, for that matter) has a physical meaning, superluminal group velocities emerge when we *extrapolate* the concept of group velocity to the region *within* the barriers. But, it is clear that inside the barriers the (evanescent) wavepacket undergoes great distortion, not sharing the same shape as the incident or transmitted ones; in fact, within the barriers the wavepacket does not even have a peak which travels from one boundary to the another [4, 10, 15]. Thus, despite the fact that the group velocity has well defined meaning for the incident (before reflections) and transmitted regions, the extrapolation of this concept to the region inside the barriers cannot be justified, and consequently there is no justification for associating this (extrapolated) group velocity with tunneling velocity.

In what concerns the dwell time, since it does does not distinguish between the transmitted and reflected channels, it is better interpreted as a cavity lifetime. Howeve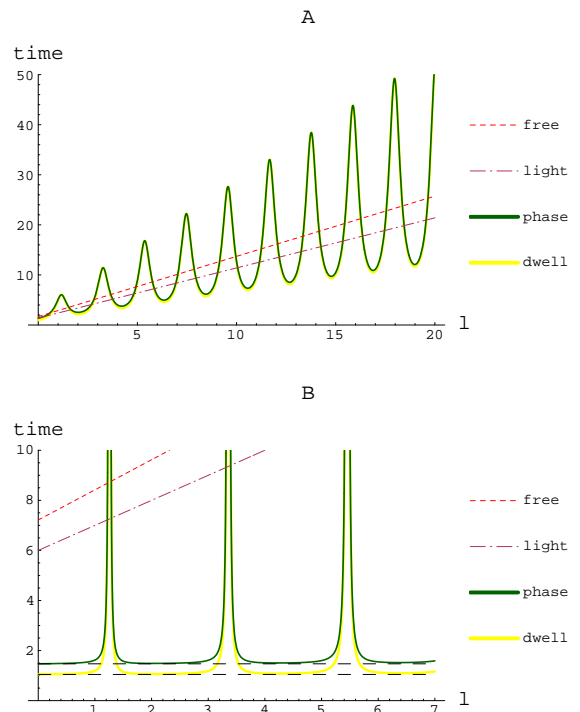r, it is important to notice that here, contrary to what happens for a Fabry-Perot cavity [4, 10], the phase and dwell times are not equal in the off-resonance case [see Figs. 2 and 3 and the limits (16)-(17)]. This prevents an immediate identification of phase time as a cavity lifetime in the present scenario.

Summarizing, in this paper we have analyzed the relativistic tunneling of a spin $1/2$ particle through two successive electrostatic potential barriers and showed that the so called generalized Hartmann effect also occurs in the realm of relativistic quantum mechanics. In addition, we obtained that the dwell time also saturates with the width of the barriers. We demonstrated that the phase and dwell times can become smaller than the light time (which implies superluminal group velocities) even before the saturated regime is obtained, and we observed that the group velocities predicted by the relativistic theory can be smaller or greater than those predicted by Schrödinger's theory, depending on the values of the parameters. We also showed that the phase and dwell times show an almost linear increasing with the separation between the barriers, and tends to saturate *only* when the barrier becomes extremely opaque. Finally, we discussed a possible interpretation of the results, favoring the argument that the group velocity cannot be interpreted as a tunneling velocity.

FIG. 3: (Color online) The phase and dwell times *vs* separation l between the barriers. $E_0 = 1.8$ and $V_0 = 1.5$. (A) $a = 0.7$. (B) $a = 3.0$. The horizontal dashed lines indicate the corresponding saturated times in the opaque limit. We used the same (natural) system of units as in the Figure 2. The peaks in both figures correspond to resonant tunneling.

APPENDIX

In this Appendix we list the explicit expressions for the phase time, τ_p, and the self-interference delay, τ_i. From

(12) and (11) we obtain

$$\tau_p = \frac{1}{\hbar c^2} \left\{ (kl)\frac{E}{k^2} - \frac{1}{k^2 q^2}\frac{h_1}{\Gamma^2 + \Delta^2} \right\} , \qquad (A.1)$$

where we have defined

$$\Gamma \equiv 8\alpha^2 \cosh(2qa) - 4(1+\alpha^2)^2 \sin^2(kl) \sinh^2(qa) , \qquad (A.2)$$

$$\Delta \equiv 4\alpha(1-\alpha^2)\sinh(2qa) + 2(1+\alpha^2)^2 \sin(2kl)\sinh^2(qa) , \qquad (A.3)$$

and

$$\begin{aligned}
h_1 \equiv & \; \Delta \left\{ 2(1+\alpha^2)\left[(1+\alpha^2)Eq^2(2kl)\sin(2kl) - 4\alpha^2 mc^2(k^2+q^2)\cos(2kl)\right]\sinh^2(qa) \right. \\
& - 4\alpha^2 mc^2(k^2+q^2)\left[(1+\alpha^2) + (3-\alpha^2)\cosh(2qa)\right] \\
& \left. + k^2(2qa)(E-V_0)\left[(1+\alpha^2)^2\cos(2kl) - (1-6\alpha^2+\alpha^4)\right]\sinh(2qa) \right\} \\
& + \Gamma \left\{ -4\alpha(1-\alpha^2)k^2(2qa)(E-V_0)\cosh(2qa) \right. \\
& + 2(1+\alpha^2)\left[(1+\alpha^2)Eq^2(2kl)\cos(2kl) + 4\alpha^2 mc^2(k^2+q^2)\sin(2kl)\right]\sinh^2(qa) \\
& \left. + \left[4\alpha(1-3\alpha^2)mc^2(k^2+q^2) - (1+\alpha^2)^2 k^2(2qa)(E-V_0)\sin(2kl)\right]\sinh(2qa) \right\} .
\end{aligned} \qquad (A.4)$$

From equations (6) and (7), the self-interference delay, as defined in (15), is given by

$$\tau_i = \frac{m}{\hbar k^2}\frac{(1+\alpha^2)}{4\alpha^3}\frac{h_2}{h_3} , \qquad (A.5)$$

with

$$\begin{aligned}
h_2 \equiv & \; \frac{1}{2}\alpha(1-\alpha^2)\sin(2kl)\sinh^2(2qa) + \alpha^2\cos^2(kl)\sinh(4qa) \\
& + \alpha(1-\alpha^2)\sin(2kl)\sinh^2(qa)\cosh(2qa) + (1-\alpha^2)^2\sin^2(kl)\sinh^2(qa)\sinh(2qa) ,
\end{aligned} \qquad (A.6)$$

$$\begin{aligned}
h_3 \equiv & \; \frac{1}{8\alpha^4}\left\{ 8\alpha^4\cosh^4(qa) + \left[1 + 6\alpha^4 + \alpha^8 - (1-\alpha^4)^2\cos(2kl)\right]\sinh^4(qa) \right. \\
& \left. + \alpha^2\left[(1-\alpha^2)^2 + (1+\alpha^2)^2\cos(2kl)\right]\sinh^2(2qa) + 2\alpha(1-\alpha^2)(1+\alpha^2)^2\sin(2kl)\sinh^2(qa)\sinh(2qa) \right\} .
\end{aligned} \qquad (A.7)$$

Finally, the dwell time is obtained from the phase time and the self-interference delay from (14).

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